A COMPARATIVE STUDY OF
CAPITAL BUDGETING AND CAPITAL RATIONING MODELS
AS AN ANALYSIS FOR CAPITAL INVESTMENT DECISIONS

A literature study
done by:
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BANDUNG
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## ABSTRACT

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The process of planning and evaluating proposals for investments is called capital budgeting. Capital budgeting decisions are important because large amounts of money are committed for long periods of time and because these types of decisions are often difficult or impossible to reverse once the funds have been committed.

This study surveys different models and techniques that are used to solve several types of capital budgeting problems. These types are of simple capital budgeting and capital rationing; under conditions of certainty and uncertainty.

The analysis helps on the difficult and critical decisions management has to make.
The field of capital investment analysis is both comprehensive and challenging. It clearly plays a vital role in assisting most business firms to achieve their various goals.

Capital budgeting is the decision area in financial management which establishes goals and criteria for investing resources in long term projects. Capital investment projects commonly include land, buildings, facilities, equipment, and the like. These assets are extremely important to the firm because, in general, nearly all of the firm's profit are derived from the use of its capital investments; these assets represent very large commitments of resources; and the funds will usually remain invested over a long period of time. The future development of the firm hinges on the selection of capital investment projects, and the decision to abandon previously accepted undertakings which turn out to be less attractive to the firm than was originally thought.

The benefits of capital projects are received over some future period, and the time element lies at the core of capital budgeting. The firm must time the start of a project to take advantage of short-term business conditions (construction costs for example vary with the stage of business cycle) and financing of the project to capitalize on trends in the money markets (such as the pattern of short and long term interest rates). In addition, the longevity of capital assets and the large outlays required for their acquisition suggest that the estimates of income and cost associated with the project to be documented for the time
are received or paid out. Moreover, investment decisions are always based upon incomplete information using forecasts of future revenues and costs. It is known from experience that such forecasts will always err on one side or the other, and the degree of error may correlate (but not always) with the duration of the projects. Short-term forecasts (1 year or less) generally display greater accuracy than long-term estimates (5 years or more). The future dimly seen entails risk, and any appraisal of a capital project, therefore, must necessarily comprehend some assessment of the risk accompanying the project. Finally, investment decisions must be matched against some future outcomes to ascertain the accuracy of the forecast and the viability of evaluating criteria. In summary, the components of capital budgeting analysis involve a forecast of the benefits and costs of the project, discounting the funds invested in the project at an appropriate rate, assessing the risk associated with the project and following up to determine if the project perform as expected.

The application of capital budgeting are many and varied. In theory, a large number of problems lend themselves to analysis by various methods. Analysis absorbs time and money, especially the more sophisticated techniques of ranking and risk management. The cost of these approaches must be justified by the perceived benefits. Theory adapts to circumstances. Conceptually appealing but costly techniques of analysis do not merit across-the-board application. Accordingly, in establishing a cutoff by size of expenditure; that is, projects requiring an investment over a specified amount will be subject to searching scrutiny; below this amount, less costly criteria of acceptance will be applied.

This study is an attempt to survey different models and techniques that are used to solve several types of capital budgeting problems. In the first place we make a
distinction between conditions of certainty and conditions of uncertainty. Another important distinction is between capital budgeting problems (no capital restriction) and capital rationing problems (capital restriction).

Chapter 2 summarizes some techniques to solve the capital budgeting problem under certainty. Five alternatives are discussed: net present value, payback period, average return on book value, internal rate of return and profitability index.

These simple techniques cannot be used in situations where the capital budget is limited. Chapter 3 investigates some techniques that can be used to solve the capital rationing problem under certainty. Simple techniques, like the ranking based on the profitability index proposed by Lorie-Savage, can be used when there is a restriction in only one period. Otherwise mathematical programming should be used: this can be linear programming, integer programming or goal programming.

Chapters 4 and 5 handle conditions of uncertainty. Techniques that can be used to solve the capital budgeting problem under uncertainty are: sensitivity analysis, break even analysis, decision trees and simulation. These techniques are described in chapter 4.

Finally, the capital rationing problem under uncertainty is discussed in chapter 5. Here, a combination of mathematical programming and simulation can be used, like Salazar and Sen proposed in their paper.

Most of the discussed techniques are illustrated with examples. For this purpose, programs available on the apple computers of the department Industrieel Beleid have been used. The existing capital budgeting package did not include a sensitivity analysis. That is why a new program was developed. This program and some examples are discussed in chapter 6.
In order to be able to perform an economic evaluation of a project's desirability, it is necessary to understand the decision rule for accepting or rejecting investment projects. Some methods which guide management in the acceptance or rejection of proposed investments are explained in this chapter.

2.1. Net present value.

This measure is a direct application of the present value concept. Its computation requires the following steps: first, choose an appropriate rate of interest. Second, compute the present value of the subtraction of the cash outlays from the investment from the cash proceeds expected from the investment. This gives the present value of the cash flows. The present value of the cash flows minus the initial investment is the net present value of the investment.

The recommended accept or reject criterion is to accept all independent investments whose net present value is greater or equal to zero and reject all investments whose net present value is less than zero.

The formula for calculating PV and NPV can simply be written as:

\[ PV = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n} \]

\[ NPV = - C_0 + PV \]
with:
\[ C_1, C_2, \ldots, C_n = \text{Cash flows of the investment.} \]
\[ r = \text{Interest rate.} \]
\[ CO = \text{The initial investment.} \]

Example:

The payment for the construction of a building is on the following schedule:

a. 100,000 down payment now. The land, worth 50,000 must also be committed now.

b. 100,000 progress payment after 1 year.

c. A final payment of 100,000 when the building is ready for occupancy at the end of the second year.

d. Despite the delay the building will be worth 400,000 when completed.

All this yields a new set of cash-flow forecasts:

<table>
<thead>
<tr>
<th>Period</th>
<th>( t=0 )</th>
<th>( t=1 )</th>
<th>( t=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land</td>
<td>- 50,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>construction</td>
<td>-100,000</td>
<td>-100,000</td>
<td>-100,000</td>
</tr>
<tr>
<td>payoff</td>
<td></td>
<td></td>
<td>+400,000</td>
</tr>
<tr>
<td>Total</td>
<td>CO=-150,000</td>
<td>C1=-100,000</td>
<td>C2=+300,000</td>
</tr>
</tbody>
</table>

If the interest rate is 7 percent, then NPV is:

\[
\text{NPV} = CO + C_1/(1+r) + C_2/(1+r)^2
\]
\[
= -150,000 - 100,000/1.07 + 300,000/(1.07)^2
\]
\[
= 18,400
\]

Since the NPV is positive, we should go ahead.

2.2. Payback.

Companies frequently require that the initial outlays on any project should be recoverable within some specified
cutoff period. The payback period of a project is found by counting the number of years it takes before cumulated forecasted cash flows equal the initial investment.

Example:
Consider projects A and B.

<table>
<thead>
<tr>
<th>Cash flows, dollars</th>
<th>Payback period, years</th>
<th>NPV at 10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project</td>
<td>CO</td>
<td>C1</td>
</tr>
<tr>
<td>A</td>
<td>-2000</td>
<td>+2000</td>
</tr>
<tr>
<td>B</td>
<td>-2000</td>
<td>+1000</td>
</tr>
</tbody>
</table>

The NPV rule tells us to reject A and accept B. With project A, we take 1 year to recover our 2000, with project B we take 2 years.

If the firm uses the payback rule with a cutoff period of 1 year, it would accept only project A. If it used the payback rule with a cutoff period of 2 or more years it would accept both A and B. The reason for the difference is that payback gives equal weight to all cash flows before the payback date and no weight at all to subsequent flows. In order to use the payback rule the firm has to decide on an appropriate cutoff date.

2.3. Average return on book value.

Some companies judge an investment project by looking at its book rate of return. To calculate book rate of return it is necessary to divide the average forecasted profits of a project after depreciation and taxes, by the average return on book value of the investment. This ratio is then
measured against the book rate of return for the firm as a whole or against some external yardstick, such as the average book rate of return for the industry.

This criterion ignores the opportunity cost of money and is not based on the cash flows of a project, and the investment decision may be related to the profitability of the firm's existing business.

Example:

The table below shows projected income statements for project A over its 3-year life. Its average net income is 2000 per year. The required investment is 9000 at t=0. This amount is then depreciated at a constant rate of 3000 per year.

<table>
<thead>
<tr>
<th>Cash flow (x 1000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project A</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Year 1</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>Revenue</td>
</tr>
<tr>
<td>Out-of-pocket cost</td>
</tr>
<tr>
<td>Cash flow</td>
</tr>
<tr>
<td>Depreciation</td>
</tr>
<tr>
<td>Net income</td>
</tr>
</tbody>
</table>

So the book value of the new investment will decline from 9000 in year 0 to zero in year 3:

<table>
<thead>
<tr>
<th>Year1</th>
<th>Year2</th>
<th>Year3</th>
<th>Year4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross book value of investment</td>
<td>9000</td>
<td>9000</td>
<td>9000</td>
</tr>
<tr>
<td>Accumulated depreciation</td>
<td>0</td>
<td>3000</td>
<td>6000</td>
</tr>
<tr>
<td>Net book value of investment</td>
<td>9000</td>
<td>6000</td>
<td>3000</td>
</tr>
<tr>
<td>Average net book value=4500</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The average net income is 2000, and the average net investment is 4500. Therefore the average book rate of return is 2000/4500 = 0.44. Project A would be undertaken if the firm's target book rate of return were less than 44%.

2.4. Internal Rate of Return.

The internal rate of return is defined as the discount rate which makes NPV=0. This means that to find the internal rate of return for an investment lasting T years, we must solve for the IRR in the following expression:

\[ NPV = C_0 + C_1/(1+IRR) + C_2/(1+IRR)^2 + \ldots + C_T/(1+IRR)^T = 0 \]

Actual calculation of IRR usually involves trial and error. The easiest way to calculate IRR, if we have to do it by hand, is to plot three or four combinations of NPV and discount rate on a graph, connect the points with a smooth line, and read off the discount rate at which NPV=0.

The rule for investment decisions on the basis of IRR is to accept an investment project if the opportunity cost of capital is less than the IRR. If the opportunity cost of capital is less than the IRR, then the project has a positive NPV when discounted at the opportunity cost of capital. If it is equal to IRR, the project has a zero NPV. And if it is greater than the IRR, the project has a negative NPV. Therefore when we compare the opportunity cost of capital with the IRR on our project, we are effectively asking whether our project has a positive NPV.

Example:
A new project has an after-tax cost of 10,000 and will result in after-tax cash inflows of 3,000 in year 1, 5,000 in year 2, and 6,000 in year 3.
NPV = \(-10,000 - \frac{5,000}{(1+IRR)} + \frac{6,000}{(1+IRR)^2} \]

By trial and error to find the IRR which gives result close to zero. Using 15% IRR, the NPV is close to zero but negative (-38). The NPV for IRR 16% is positive (146). The actual IRR is between 16% and 17% and may be found using linear interpolation.

Example: NPV versus IRR
Consider two projects A and B which are mutually exclusive. The cost of capital is 10%.

<table>
<thead>
<tr>
<th>Project</th>
<th>year 0</th>
<th>year 1</th>
<th>year 2</th>
<th>year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-1,000</td>
<td>505</td>
<td>505</td>
<td>505</td>
</tr>
<tr>
<td>B</td>
<td>-11,000</td>
<td>5,000</td>
<td>5,000</td>
<td>5,000</td>
</tr>
</tbody>
</table>

The NPVs and the IRRs of the two projects are:

<table>
<thead>
<tr>
<th>IRR</th>
<th>NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>24%</td>
</tr>
<tr>
<td>B</td>
<td>17%</td>
</tr>
</tbody>
</table>

If the firm uses the NPV criterion, project B will be chosen. However if the firm uses the IRR criterion project A will be preferred.

Now, we consider the incremental cash flow:

<table>
<thead>
<tr>
<th>Project</th>
<th>year 0</th>
<th>year 1</th>
<th>year 2</th>
<th>year 3</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-A</td>
<td>-10,000</td>
<td>4,495</td>
<td>4,495</td>
<td>4,495</td>
<td>16.58%</td>
</tr>
</tbody>
</table>

The IRR on this incremental cash flow is 16.58%, and given a 10% cost of capital, this represents a profitable
opportunity and should be accepted. This results in a total cash flow to the firm of $A + (B-A) = B$. Thus, the IRR rule when used properly (i.e. on incremental basis) leads the firm to prefer project B, but this is precisely the project which has the higher NPV.

2.5. Profitability index.

The profitability index is the present value of the forecasted cash flows divided by the initial investment

\[ \text{Profitability index} = \frac{\text{present value}}{\text{investment}} = \frac{PV}{CO} \]

The profitability index rule tells us to accept all projects with an index greater than 1. If the profitability index is greater than 1, the present value is greater than the initial investment, and so the project must have a positive NPV. The profitability index, therefore, leads to exactly the same decisions as NPV.

For many people the profitability index is more intuitively appealing than the NPV criterion. The statement that a particular investment has a NPV of say $20$ is not sufficiently clear to many people who prefer a relative measure of profitability. By adding the information that the project's initial outlay (CO) is $100$, the profitability index \( \frac{120}{100} = 1.2 \) provides a meaningful measure of the project's relative profitability in more readily understandable terms. It is then only a small step to convert the index of 1.2 to 20%.

However once again problems can arise when mutually exclusive alternatives are considered. The profitability index may be useful for exposition, it should not be used as a measure of investment worth for projects of differing size when mutually exclusive choices have to be made.
Example:
Consider the following two projects:

<table>
<thead>
<tr>
<th>Project</th>
<th>PV at 10%</th>
<th>Profitability index</th>
<th>NPV at 10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>182</td>
<td>1.32</td>
<td>82</td>
</tr>
<tr>
<td>L</td>
<td>13,636</td>
<td>1.36</td>
<td>3.636</td>
</tr>
</tbody>
</table>

Both are good projects, as the profitability index correctly indicates. But suppose the projects are mutually exclusive. We should take L, the project with the higher NPV. Yet the profitability index gives K the higher ranking.

We can always solve such problems by looking at the profitability index on the incremental investment.

Example:

<table>
<thead>
<tr>
<th>Project</th>
<th>PV at 10%</th>
<th>Profitability index</th>
<th>NPV at 10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>L-K</td>
<td>13,454</td>
<td>1.36</td>
<td>3554</td>
</tr>
</tbody>
</table>

The profitability index on the additional investment is greater than 1, so L is the better project.

2.6. Application of the investment analysis program.

At the department, a program has been developed to solve capital budgeting problems under certainty. The following criteria are considered: net present value, profitability index, internal rate of return and
(discounted) payback period.

Figure 2.6 is based on example 2.1 (tax effects are also considered now) and illustrates input and output data. The program allows several depreciation schemes and different types of revenues and expenses. As output is concerned, not only the investment criteria are given, but also a detailed picture of the periodic cash flows.
INPUT DATA
**********

1. GENERAL DATA

| PROJECT LIFETIME | = | 2 |
| TAX RATE (%)     | = | 50 |
| INTEREST RATE (%)| = | 7 |
| INFLATION RATE (%)| = | 0 |
| ACTUALISATION RATE (%)| = | 7 |

2. INVESTMENT AND DEPRECIATION DATA

| TIME OF INVESTMENT | LAND | CONSTRUCT |
| AMOUNT              | 50000 | 100000 |
| RESIDUAL VALUE      | 0    | 0      |
| DEPRECIATION DURATION| 2    | 2      |
| DEPRECIATION METHOD | V    | V      |

3. REVENUES

<table>
<thead>
<tr>
<th>PAYOFF</th>
</tr>
</thead>
<tbody>
<tr>
<td>TYPE</td>
</tr>
<tr>
<td>V</td>
</tr>
<tr>
<td>PERCENT OR LINEAR INCR</td>
</tr>
<tr>
<td>SPECIFIC INFLATION</td>
</tr>
<tr>
<td>REVENUE FIRST PERIOD</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>PAYOFF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>400000</td>
</tr>
</tbody>
</table>

4. EXPENSES

| CONSTR |
| TYPE |
| V    |
| PERCENT OR LINEAR INCR |
| SPECIFIC INFLATION |
| EXPENSE FIRST PERIOD |

<table>
<thead>
<tr>
<th>EXPENSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>100000</td>
</tr>
</tbody>
</table>
5. PERIODIC DEPRECIATIONS

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>LAND</th>
<th>CONSTRUCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
## Output Data

### 1. Evaluation Criteria

- **Project Lifetime**: 2
- **Tax Rate**: 50
- **Actualisation Rate**: 7

<table>
<thead>
<tr>
<th>Without Taxes</th>
<th>With Taxes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net Present Value</td>
<td>18574</td>
</tr>
<tr>
<td>Profitability Index</td>
<td>1.124</td>
</tr>
<tr>
<td>Internal Rate of Return</td>
<td>11.96</td>
</tr>
<tr>
<td>Payback Period</td>
<td>1.929</td>
</tr>
</tbody>
</table>

### 2. Periodic Flows

<table>
<thead>
<tr>
<th>Period</th>
<th>Revenue</th>
<th>Expense</th>
<th>Operating</th>
<th>Depreciation</th>
<th>Profit</th>
<th>Taxes</th>
<th>After Tax</th>
<th>Cash Flow</th>
<th>Investment</th>
<th>Salvage</th>
<th>Salaries</th>
<th>Taxes</th>
<th>Net Cash</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>150000</td>
<td>0</td>
<td>0</td>
<td>-15000</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>100000</td>
<td>-100000</td>
<td>0</td>
<td>0</td>
<td>-10000</td>
<td>-50000</td>
<td>-50000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-50000</td>
</tr>
<tr>
<td>3</td>
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### 3. Periodic Revenues

<table>
<thead>
<tr>
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<th>Revenue</th>
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### 4. Periodic Expenses

<table>
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</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>100000</td>
</tr>
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</table>
Chapter 3.

CAPITAL RATIONING UNDER CERTAINTY

3.1. Definition

The capital budgeting problem involves the allocation of scarce capital resources among competing economically desirable projects, not all of which can be carried out due to a capital (or other) constraint. This problem is referred to as capital rationing.

It is often that the restriction on the supply of capital reflects non-capital constraints or bottlenecks within the firm. For example, the supply of key personnel to carry out the projects maybe severely limited, thereby restricting the dollar amount of feasible investment. Similarly, consideration of management time may preclude the adoption of programs beyond some level. This should more properly be called labor or management constraints rather than capital constraints. These restrictions limit or fix the maximum amount of investment which can be undertaken by the firm.

Many firms' capital constraints are soft. They reflect no imperfection in the capital market. Instead they are provisional limits adopted by management as an aid to financial control. Soft rationing should never cost the firm anything. If capital constraints become tight enough to hurt, then the firm raises more money and loosens the constraints. But what if it cannot raise more money—what if it faces hard rationing?

Hard rationing implies market imperfection—a barrier between the firm and capital markets. If that barrier also implies that the firm's shareholders lack free access to a well-functioning capital market, the very foundation of net present value crumble.
3.2. Structure.

The entire discussion of methods of capital rationing rests on the proposition that the wealth of a firm's shareholder is highest if the firm accepts every project that has a positive net present value. Suppose, however, that there are limitations on the investment program that prevent the company to undertake all such projects. If this is the case, we need a method of selecting the package of projects that is within the company's resources yet gives the highest possible net present value.

We will survey several techniques which are applied to the capital budgeting problem setting, under conditions of certainty and under conditions of risk:

- Capital rationing under conditions of certainty:

  Lorie and Savage proposed profitability index techniques for the single period case and the multi-period case. Weingartner formulates the capital rationing problem first as an LP then as an integer programming model.

  Since these pioneering works, there have been many advances in the area of mathematical programming applied to the capital budgeting problem, e.g. goal programming model.

- Capital rationing under conditions of uncertainty (see chapter 5).

3.3. Profitability Index.

Lorie and Savage proposed a solution to the capital rationing problem for the single period case and the multi-period case, on the assumption that:
1. The timing and magnitude of the cash flows of all projects are known at the outset with complete certainty.
2. The cost of capital is known (independent of the investment decisions under consideration) so that the present value of all the projects are also given.
3. All projects are strictly independent, i.e. the execution of one project does not affect the costs and benefits of some other project.

Under these simplifying assumptions our problem becomes one of how to select among projects which have positive net present values.

For a single period case, the solution is to rank all projects by the profitability index and then select from the top of the list until the budget is exhausted. The procedure is simple and easily understood. However, it depends on a set of very limiting and unrealistic assumptions: namely that cash flows are known, the cost of capital is independent of the investment decision, mutually exclusive projects are ruled out; and all outlays occur in a single period of time so that the budget constraint applies to a single period.

Lorie and Savage drop the latter restriction by also considering the selection process in which the budget limitation occurs in more than one period. Like also many seminal articles in finance and economics it is the question they asked, and not the particular solution which they proposed, which is of lasting interest.

Example:
Lorie and Savage consider the following example:
Suppose the budget is 100. This means projects 3, 4, 6, 5, 1, are selected, and the budget used is 98.

3.4. Mathematical Programming: Linear Programming.

By the linear programming (LP) the firm can examine very large-scale choice problem in which the number of alternatives is virtually unlimited in the relevant range. It helps allocating and evaluating scarce resources and provides decision makers with some very insightful information regarding the marginal value of resources.

The LP formulation of capital rationing is as follows:

\[
\begin{align*}
\text{max NPV} & = \sum_{j=1}^{N} b_j x_j \quad (1) \\
\text{s.t} & \sum_{j=1}^{N} c_{jt} x_j \leq x_t \quad t=1,2,...,T \quad (2) \\
& 0 \leq x_j \leq 1 \quad (3)
\end{align*}
\]

\( x_j \) = percent of project \( j \) that is accepted  
\( b_j \) = NPV of project \( j \) over its useful life
\[ C_{jt} = \text{cash outflow required by project } j \text{ in year } t \]
\[ K_t = \text{budget available in year } t \]

The following aspects should be noted about the problem formulation above:
1. The \( x_j \) decision variables are assumed to be continuous—that is, partial projects are allowed in the LP formulation.
2. The usual nonnegativity constant of LP is modified as shown in equation (3) to also show an upper limit for each project.
3. It is assumed that all the input parameters — \( b_j, c_j, K_t \), are known with certainty.
4. The \( b_j \) parameter shows the NPV of project over its useful life, where all cash flows are discounted at the cost of capital, which is known with certainty.

The marginal value to the firm of the budget constraints are obtained by comparing the total NPV with and without an extra dollar of resources. In LP such values are referred to as dual variables or shadow prices. They represent the opportunity costs of using a unit of the firm's resources.

The dual LP formulation for the capital rationing problem is as follows:

\[
\begin{align*}
\text{Min} & \quad \sum_{t=1}^{T} \rho_t K_t + \sum_{j=1}^{N} \gamma_j \\
\text{s.t} & \quad \sum_{t=1}^{T} \rho_t C_{jt} + \mu_j \geq b_j, \quad j = 1, 2, \ldots, N \\
& \quad \rho_t, \mu_j \geq 0, \quad t = 1, 2, \ldots, T \text{ and } j = 1, 2, \ldots, N
\end{align*}
\]

where:
\( \rho_t = \text{dual decision variable which represents the cost associated with resource of type } t. \)
\( y_j = \) dual decision variable associated with project \( j \).

The dual formulation has important implications for the financial manager. Namely, the dual LP and its optimal solution provide valuable information for both planning and control functions in the capital budgeting decision process. This optimal solution gives the shadow prices for the accepted and rejected projects.

These values enable the decision maker to rank all projects according to their relative attractiveness.

For accepted projects the shadow prices are found under the slack variables \( s_k \) for the following constraint:

\[ x_j + s_k = 1 \]

The shadow prices are computed using the following expression:

\[ y_j = b_j - \sum_{t=1}^{T} \rho_t^* C_{jt} \]

- \( y_j \) = shadow price associated with accepted project \( j \), shown under the slack variable associated with project \( j \)
- \( b_j \) = NPV for project \( j \) shown in the objective function.
- \( \rho_t^* \) = shadow price in the optimal solution associated with each resource \( t \) which is required to accept a project. (shown under the slack variable associated with the corresponding resources).
- \( C_{jt} \) = quantity of resources of type \( t \) required by project \( j \).

The shadow prices \( y_j \) may very well give a ranking for the projects which differs from that given by any of the simple models as payback, NPV, IRR, or the profitability index. Such differences in ranking will exist because the latter models look at projects independently.
The shadow prices show interrelationships among projects by means of the budget constraints. They evaluate the projects at cost of capital \((p_x)\) that is implied by the optimal use of resources of type \(t\).

For the rejected projects the shadow prices are computed in an analogous way as for accepted project.

\[
\mu_j = \sum_{t} p_x C_{jt} - b_j
\]

\(\mu_j\) = shadow prices associated with rejected project \(j\) (shown in the objective function row in the column for \(x_j\)).

The \(\mu_j\) value shows the amount by which the objective function would decrease if the firm were forced to accept the unattractive project \(j\). If such projects were accepted, this would mean that the scarce capital budget dollars would be used in a suboptimal way, since the opportunity cost associated with the cash outflows \((-\sum p_x C_{jt})\) exceeds the present value of the benefits generated by the projects.

It should be mentioned that the values must be zero for all projects that are accepted (including partially accepted projects) because the benefits of these projects must justify the cash outlays in the various periods of the planning horizon \((C_{jt})\) when they are evaluated at the implied cost of capital \((p_x)\) when the budgets each year are used in an optimal way.

Example:
Lorie and Savage nine-project problem.
<table>
<thead>
<tr>
<th>Project</th>
<th>NPV</th>
<th>Cash outflow in period 1</th>
<th>Cash outflow in period 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
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<td>3</td>
</tr>
<tr>
<td>9</td>
<td>12</td>
<td>18</td>
<td>3</td>
</tr>
</tbody>
</table>

Budget available \( \sum C_{ij} x_{ij} \leq 50 \)

The LP formulation is as follows:

Max \( \text{NPV} = 14x_1 + 17x_2 + 17x_3 + 15x_4 + 40x_5 + 12x_6 \)
+ \( 14x_7 + 10x_8 + 12x_9 \)

s.t.

\( 12x_1 + 54x_2 + 6x_3 + 6x_4 + 30x_5 \)
+ \( 6x_6 + 48x_7 + 36x_8 + 18x_9 + S_1 = 50 \). budget constraint

year 1

\( 3x_1 + 7x_2 + 6x_3 + 2x_4 + 35x_5 \)
+ \( 6x_6 + 4x_7 + 3x_8 + 3x_9 + S_2 = 20 \). budget constraint

year 2

\( x_i + S_3 = 1 \) \( x_i + S_4 = 1 \) \( x_7 + S_5 = 1 \) Upper limits

\( x_1 + S_6 = 1 \) \( x_7 + S_7 = 1 \) \( x_7 + S_9 = 1 \) on project

\( x_3 + S_{10} = 1 \) \( x_6 + S_8 = 1 \) \( x_7 + S_{11} = 1 \) acceptance

\( x_j, S_i \geq 0 \) \( i = 1, 2, \ldots, 11 \). Non negativity

\( j = 1, 2, \ldots, 9 \) constraint

The problem has been solved with a package on Apple II developed at the department.
Solution

.MAX PROBLEM  TWO-PERIOD LP
*CONSTRAINTS  11 (<30) L=11 G= 0 E= 0
*VARIABLES  20 (<80) D= 9 S=11 A= 0

OPTIMAL SOLUTION  70.2727273

VALUE OF X1  =>  1
VALUE OF X2  =>  0
VALUE OF X3  =>  1
VALUE OF X4  =>  1
VALUE OF X5  =>  0
VALUE OF X6  =>  .969697
VALUE OF X7  =>  .045455
VALUE OF X8  =>  0
VALUE OF X9  =>  1

CONSTRAINT ANALYSIS

<table>
<thead>
<tr>
<th>CONSTRAINT</th>
<th>ACTIV?</th>
<th>MARGINAL VALUE</th>
<th>LEVEL OF SLACK (S1)</th>
</tr>
</thead>
<tbody>
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<tr>
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</tr>
<tr>
<td>UPPER-2</td>
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<td>UPPER-3</td>
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<tr>
<td>UPPER-9</td>
<td>YES</td>
<td>3.954545</td>
<td>0</td>
</tr>
</tbody>
</table>
We see that the basic variables, that is, the variables that are equal to a positive value in the optimal solution are: \( x_1, x_3, x_4, x_5, x_6 \) (column value of \( x \)) and \( S_4, S_7, S_9, S_{10} \) (column level of slack).

Any of the variables in the problem which are equal to zero, are in fact, non basic variables in the optimal solution. Thus, \( x_1 = x_3 = x_5 = 0 \), which shows that these three projects should be completely rejected; in addition \( S_4 = S_7 = 0 \), shows that the entire budget of 50 in year 1 and 20 in year 2 has been spent on the six projects that have been designated for acceptance. Further, \( S_9 = S_{10} = 0 \), since the project corresponding to these slack variables have been 100% accepted.

These projects require the use of the entire budget in both years and generate the maximum objective function value of 70,272,727,3, which is the NPV of the accepted projects.

The marginal value of constraint number 1 (the shadow price of \( S_4 \)) is 0.136364 which means that the objective function would increase by this amount (0.136364) for each dollar of additional budget they could obtain. This constraint is active because any changes in the constraint can change the objective function.

The same explanation can be used for the other constraints.

LP models seem tailor-made for solving capital budgeting problems when resources are limited. Why then are they not universally accepted either in theory or in practice? One reason is that these models are often not cheap to use. LP is considerably cheaper in terms of computer time, but it cannot be used when large, indivisible projects are involved.

As with any sophisticated long-range planning tool there is the general problem of getting good data. It is not just worth applying costly, sophisticated methods for poor
data. Furthermore, these models are based on the assumption that all future investment opportunities are known. In reality, the discovery of investment ideas is an unfolding process.

Before we leave the area of linear programming a noteworthy controversy relative to the appropriate LP formulation should be mentioned. Under capital rationing, the appropriate discount rate to use in determining the net present values of projects under consideration cannot be determined until the optimal set of projects is determined, so that the size of the capital budget is ascertained as well as the sources of the subsequent financing and hence the cost of capital (or the appropriate discount rate to use in calculating NPVs). This is a simultaneous problem wherein the firm should concurrently determine through an iterative mathematical programming process both the optimal set of capital projects and the optimal financing package, with its associated cost of capital to be used in the discounting process.

Weingartner suggested an operational approach; his model assumes that all shareholders have the same linear utility preferences for consumption. And it makes the period-by-period utilities independent of one another. He then proposes a more operational model, which maximizes the dividends to be paid in a terminal year, where throughout the planning horizon dividend are nondecreasing and can be required to achieve a specified annual growth rate. Over the past decade, several other authors have jumped into the controversy, each suggesting his own reformulation.

The Weingartner's basic horizon model is as follows:

\[
\begin{align*}
\text{Max} & \quad Z = \sum J \lambda_j x_j + v_\tau - v_\tau \\
\text{s.t.} & \quad \sum J a_{ij} x_j + v_\tau - v_\tau \leq D_i \quad (a)
\end{align*}
\]
\[
\sum_j a_{ij} x_j - (1+r)v_{e-1} + v_x + (1+r)w_{e-1} - w_x \leq D_t \quad t = 2, \ldots, T
\]  
(b)

\[
0 \leq x_j \leq 1 \quad j = 1, \ldots, n
\]  
(c)

\[
v_x, w_x \geq 0 \quad t = 1, \ldots, T
\]  
(d)

where:
\( \tilde{a}_j \) = value of all cash flows of project \( j \) subsequent to the horizon discounted to the horizon at the market rate of interest, \( r \).
\( x_j \) = fraction of project \( j \) accepted.
\( T \) = horizon year.
\( v_t \) = amount available for lending in period \( t \).
\( w_t \) = amount borrowed in period \( t \).
\( a_{ij} \) = cash flow in period \( t \) from project \( j \). (pos=expenditure/outflow, neg=revenue/inflow)
\( D_t \) = anticipated cash throw-off in period \( t \).

The dual of the basic horizon model is as follows:

\[
\text{Min } \sum_{t=1}^T \rho_t D_t + \sum_{j=1}^n \mu_j
\]  
s.t.

\[
\sum_{t=1}^T \rho_t a_{ij} + \mu_j \leq \tilde{a}_j \quad j = 1, \ldots, n.
\]  
(1)

\[
\rho_t \geq 1
\]  
(2)

\[
\rho_t \geq -1
\]  
(3)

\[
\rho_{e-1} - (1+r)\rho_e \geq 0 \quad t = 2, \ldots, T
\]  
(4)

\[
-\rho_{e-1} + (1+r)\rho_e \geq 0 \quad t = 2, \ldots, T
\]  
(5)

\[
\rho_t, \mu_t \geq 0
\]  
(6)
The following derivations help to understand the meaning of the dual variables:

(2) and (3): \( \rho_t^* = 1 \)  

(4) and (5): \( \frac{\rho_t^* - 1}{\rho_t} = (1+r) \)  

(7) and (8): \( \rho_t^* = (1+r) \rho_{t+1}^* \)

\[ \begin{align*}
&= (1+r)^2 \rho_{t+2}^* \\
&= (1+r)^{T-t} \rho_{t+T-t}^* \\
&= (1+r)^{T-t}
\end{align*} \]

So, one can see that \( \rho_t^* \) is the compounded rate of interest which expresses the yield at the horizon of an additional dollar in year \( t \).

When a project is fully accepted, that is, \( x_j^* = 1 \), the corresponding restriction of the form of (a) becomes:

\[ \mu_j^* = \hat{a}_j - \sum_{t=1}^{T} \rho_t^* a_{t,j} \]

\[ \begin{align*}
&= \hat{a}_j + \sum_{t=1}^{T} (-a_{t,j})(1+r)^{T-t} \\
&= \hat{a}_j - \sum_{t=1}^{T} \rho_t^* a_{t,j}
\end{align*} \]

This is the value of the project at time \( T \). This is the same as saying that the NPV of the project should be positive.

For the rejected project we know that \( x_j^* = 0 \), and also \( x_j^* \neq 1 \), and therefore \( \mu_j^* = 0 \).

\[ 0 = \mu_j^* = \hat{a}_j - \sum \rho_t^* a_{t,j} \]

This is the same as saying that the NPV is smaller than 0.
3.5. Integer programming applied to the capital rationing problem.

The main motivations for the use of IP in the capital rationing problem setting are the following:
1. Difficulties imposed by the acceptance of partial projects in LP are eliminated.
2. All the project interdependencies can be formally included in the constraints of ILP, while the same is not true for LP due to the possibility of accepting partial projects.

In using the simple capital budgeting models (NPV, IRR, PI, etc) it is assumed that all the investment projects are independent each other (i.e. that project cash flows are not related to each other and do not influence or change one another if various projects are accepted). In using ILP, virtually any project dependencies (mutually exclusive, prerequisite, complementary) can be incorporated into the model.

The general ILP formulation for the capital rationing:

\[
\text{Max } \text{NPV} = \sum_{j=1}^{N} b_j x_j \\
\text{st} \quad \sum_{j=1}^{N} C_{jt} x_j \leq K_t \quad t=1,2,\ldots,T \\
\quad x_j = [0,1] \quad j=1,2,\ldots,N
\]

There are three types of project dependencies:

a. Mutually exclusive projects.
   A set of projects wherein the acceptance of one project in the set includes the simultaneous acceptance of any other project in the set. It is incorporated in the model by the following constraint:
\[ \sum_{j \in J} x_j \leq 1 \]

J = set of mutually exclusive projects under consideration

\( j \in J \) = that project j is an element of the set of mutually exclusive project J.

This constraint states that at most one project from set J can be accepted; this means that the firm could choose not to accept any project from set J. On the other hand, if it was necessary to select one project from the set, the above constraint would appear as a strict equality:

\[ \sum_{j \in J} x_j = 1 \]

b. Prerequisite/contingent projects.

Two or more projects wherein the acceptance of one project (A) necessitates the prior acceptance of some other project(s) (Z):

\[ x_A \leq x_Z \]

If project A cannot be accepted unless project Z is accepted, we could say that project Z is a prerequisite project for acceptance of project A; alternatively, we could say that the acceptance of project A is contingent upon the acceptance of project Z. However project Z can be accepted on its own and project A rejected.

c. Complementary projects.

Wherein the acceptance of one project enhances the cash flows of one or more other projects.

There are two complementary projects, 7 and 8. Either of these projects can be accepted in isolation. However, if both are accepted simultaneously: the cost will be reduced by say 10%. The net cash inflow will be increased by 15%. To handle the problem, a new project (project 78) would be constructed. The constraint below is needed to preclude acceptance of both projects 7 and 8 as well as
78; because the latter is the composite project consisting of the two former projects.

\[ x_1 + x_8 + x_{18} \leq 1 \]

The shortcomings of the ILP:

a. ILP is probably only feasible for small to medium size capital budgeting problems (25 constraints and 100 projects). We have to think whether the price that has to be paid in moving to ILP from LP is worth the benefit gained.

b. Meaningful shadow prices (which show the marginal change in the value of the objective function for an incremental change in the right hand side of various constraints) are not available in ILP. That is, many of the constraints on ILP problem which are not binding on the optimal integer solution will be assigned shadow prices of zero, which indicates that these resources are "free goods". In reality this is not true since the objective function would clearly decrease if the availability of such resources were decreased.

Example:
Consider the following 15 projects:
### Cash outflows

<table>
<thead>
<tr>
<th>Project</th>
<th>C1j</th>
<th>C2j</th>
<th>C3j</th>
<th>NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
<td>80</td>
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</tr>
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<td>44</td>
</tr>
<tr>
<td>5</td>
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<td>42</td>
<td>0</td>
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</tr>
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<td>6</td>
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<td>52</td>
<td>20</td>
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</tr>
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<td>7</td>
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<td>18</td>
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</tbody>
</table>

**Budget constraints:** \( \sum C1jXj \leq 300; \sum C2jXj \leq 540; \sum C3jXj \leq 380 \)

**Project interrelationships:**

1. Of the set of projects 3, 4, and 8, at most two can be accepted.
2. Projects 5 and 9 are mutually exclusive, but one of the two must be accepted.
3. Project 6 cannot be accepted unless both projects 1 and 14 are accepted.
4. Project 1 can be delayed 1 year, the same cash outflows will be required, but the NPV will drop to 22.
5. Projects 2 and 3 and projects 10 and 13 can be combined into complementary or composite projects wherein total cash outflows will be reduced by 10% and NPV increased by 12% compared to the total of the separate projects.
6. At least one of the two composite projects above must be accepted

Solution:

Maximize NPV:

\[ \text{Maximize NPV :} \]
\[ 24X_1 + 38X_2 + 40X_3 + 44X_4 + 20X_5 + 64X_6 + 27X_7 \]
\[ + 48X_8 + 18X_9 + 29X_{10} + 32X_{11} + 38X_{12} + 25X_{13} + 18X_{14} \]
\[ + 28X_{15} + 22X_{16} + 87.36X_{17} + 60.48X_{18} \]

subject to

\[ 40X_1 + 50X_2 + 45X_3 + 60X_4 + 68X_5 + 75X_6 + 38X_7 \]
\[ + 24X_8 + 12X_9 + 6X_{10} + 85.5X_{17} + 5.4X_{18} \leq 300 \]

\[ 80X_1 + 65X_2 + 55X_3 + 48X_4 + 42X_5 + 52X_6 + 90X_7 \]
\[ + 40X_8 + 66X_9 + 88X_{10} + 72X_{11} + 50X_{12} + 34X_{13} \]
\[ + 22X_{14} + 12X_{15} + 40X_{16} + 108X_{17} + 109.8X_{18} \leq 540 \]

\[ 5X_1 + 10X_3 + 8X_4 + 20X_6 + 14X_7 + 70X_8 + 20X_9 \]
\[ + 17X_{10} + 60X_{11} + 80X_{12} + 56X_{13} + 76X_{14} + 104X_{15} \]
\[ + 80X_{16} + 13.5X_{17} + 67.5X_{18} \leq 380 \]

\[ X_1 + X_4 + X_8 \leq 2 \]
\[ X_5 + X_9 = 1 \]
\[ 2X_6 \leq X_1 + X_4 \quad \text{either constraint (g) or} \]
\[ 2X + \frac{1}{2} X_{16} + X_{14} \quad \text{(h) must be satisfied} \]
\[ X_1 + X_{16} \leq 1 \]
\[ X_2 + X_3 + X_{17} \leq 1 \]
\[ X_{16} + X_{13} + X_{18} \leq 1 \]
\[ X_{17} + X_{18} \geq 1 \]
\[ X_i = [0,1] \quad i = 1, 2, \ldots, 18 \]

\[ X_{46} \] is a decision variable to denote the delay of project for 1 year
$X_{17}$ is a decision variable to denote the acceptance of the composite of projects 2 and 3.

$X_{18}$ is a decision variable to denote the acceptance of the composite of projects 10 and 13.

3.6. Goal Programming applied to the capital rationing problem.

Under conditions of certainty and perfect capital markets, the selection of the set of capital projects that maximize NPV will guarantee maximization of shareholder's wealth or utility.

However, if capital market imperfections exist (such as capital rationing, differences in lending and borrowing rates, etc) then the maximization of NPV may very well not lead to the maximization of shareholders' wealth. In addition, investors and managers are interested in and motivated by several objectives as follows: growth and stability of earnings and dividend per share; growth in sales, market share and total assets; growth and stability of reported earnings or accounting profit; favorable use of financial leverage; and return on sale, equity and operating assets.

Thus only a model that incorporates multiple criteria as objective can be a robust, yet operational, representation of the pluralistic design environment found in real-world capital budgeting problem setting.

Goal programming is such a multi-criteria model. It is capable of handling decision situations which involve a single, goal or multiple goals. Frequently the multiple goals of a decision maker must be measured by different standards. Often one goal or set of goals can be achieved only at the expense of other goals or set of goals. Goal programming allows for an ordinal ranking of goals so that
low priority goals are considered only after higher priority goals have been satisfied to the fullest extent possible.

The general Goal programming (GP) model.

The basic assumptions underlying the LP model are also valid for the GP model. The significant difference in structure is that the GP does not attempt to maximize or minimize the objective criterion directly as does the LP model. It rather, seeks to minimize the deviations between the desired goals and the actual results according to the priorities assigned. The general GP model is expressed as follows:

$$\text{Min } \sum (d_i^+ + d_i^-)$$

s.t. $$\sum_{j=1}^{n} a_{ij} x_j - d_i^+ + d_i^- = b_i \quad i = 1, \ldots, n$$

$$x_j, d_i^+, d_i^- \geq 0$$

where
- $d^+$ represents degree of over achievement of a goal.
- $d^-$ represents degree of under achievement of a goal.
- $i = \text{goals}$
- $b_i = \text{target value}$.

GP will move the values of the deviational variables as close to zero as possible within the environmental constraints and the goal structure outlined in the model.

Example: GP Formulation and Solution of the Lorie-Savage Problem

The same firm that was evaluating the nine projects now feels that the NPV objective should be supplemented with four other goals which reflect the short-run attractiveness of the projects. Specifically, the firm feels that stability and growth in sales as well as net income are very important vehicles to assist the firm in maximizing shareholders' wealth.
The following table shows the contribution that each of the projects will make to net income and sales growth in the next 2 years.

<table>
<thead>
<tr>
<th>Net Income</th>
<th>Sales Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project</td>
<td>Year 1</td>
</tr>
<tr>
<td>1</td>
<td>2.0</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
</tr>
<tr>
<td>3</td>
<td>1.6</td>
</tr>
<tr>
<td>4</td>
<td>1.2</td>
</tr>
<tr>
<td>5</td>
<td>3.0</td>
</tr>
<tr>
<td>6</td>
<td>1.1</td>
</tr>
<tr>
<td>7</td>
<td>1.5</td>
</tr>
<tr>
<td>8</td>
<td>1.2</td>
</tr>
<tr>
<td>9</td>
<td>1.3</td>
</tr>
</tbody>
</table>

The firm wants to achieve net income levels of 8 and 16, respectively, in years 1 and 2, and sales growth of 0.08 in each year, as well as to maximize NPV.

Two objectives are considered:

1. Placing the net income goals on priority 1, the sales goals on priority 2, and the NPV goal on priority 3; on the first two priority levels the year 1 goals should be weighted twice as importantly as the year 2 goals.

2. All five goals placed on priority level 1 but with relative weights of 10 for net income in year 1, 2 for net income in 2, 5 for sales growth in year 1, and 2 for sales growth in year 2, and 1 for NPV.
Solution:
The GP formulation is as follows:

\[12X_1 + 54X_2 + 6X_3 + 6X_4 + 30X_5 + 6X_6 + 48X_7 + 36X_8 + 18X_9 + S_1 = 50\]

**Budget constraint year 1**

\[3X_1 + 7X_2 + 6X_3 + 2X_4 + 35X_5 + 6X_6 + 4X_7 + 3X_8 + 3X_9 + S_2 = 20\]

**Budget constraint year 2**

\[
\begin{align*}
X_1 + S_3 & = 1 \\
X_4 + S_6 & = 1 \\
X_7 + S_9 & = 1 \\
X_2 + S_7 & = 1 \\
X_5 + S_8 & = 1 \\
X_8 + S_{10} & = 1 \\
X_3 + S_5 & = 1 \\
X_6 + S_8 & = 1 \\
X_9 + S_{11} & = 1 \\
\end{align*}
\]

**Upper limits on project**

\[j = 1, 2, \ldots, 9\]

**Nonnegativity constraint**

\[X_j, S_i \geq 0 \quad i = 1, 2, \ldots, 11\]

**Net income year 1**

\[2X_1 + 2X_2 + 1.6X_3 + 1.2X_4 + 3X_5 + 1.1X_6 + 1.5X_7 + 1.2X_8 + 1.3X_9 + d_1 - d_1 = 8\]

**Net income year 2**

\[4X_1 + 4.2X_2 + 2.5X_3 + 2.8X_4 + 5X_5 + 1.4X_6 + 3X_7 + 1.8X_8 + 2.4X_9 + d_2 - d_2 = 16\]

**Sales growth year 1**

\[0.02X_1 + 0.01X_2 + 0.02X_3 + 0.01X_4 + 0.03X_5 + 0.01X_6 + 0.01X_7 + 0.01X_8 + 0.01X_9 + d_3 - d_3 = 0.08\]

**Sales growth year 2**

\[0.03X_1 + 0.03X_2 + 0.02X_3 + 0.02X_4 + 0.04X_5 + 0.01X_6 + 0.02X_7 + 0.015X_8 + 0.018X_9 + d_4 - d_4 = 0.08\]

**NPV**

\[14X_1 + 17X_2 + 17X_3 + 15X_4 + 40X_5 + 12X_6 + 14X_7 + 10X_8 + 12X_9 + d_5 - d_5 = 40\]
Notice again that the goal level for the NPV goal is an arbitrary achievable value.

The two objective functions are as follows:

1. Minimize weighted deviations =  
   \[ P_1(2d^- + d_2^+ ) + P_2(2d_3^+ + d_4^- ) + P_3( d_5^- + d_6^+ ) \]

2. Minimize weighted deviations =  
   \[ P_1(10d^- + 2d_2^+ + 5d_3^- + 2d_4^- + d_5^- - d_6^+ ) \]

The optimal solution for the two objective functions are:

Objective function 1:

<table>
<thead>
<tr>
<th>Project acceptance</th>
<th>Goal levels achieved</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 = 1.0000 )</td>
<td>7.231 Net income year 1</td>
</tr>
<tr>
<td>( X_2 = 0.0426 )</td>
<td>13.209 Net income year 2</td>
</tr>
<tr>
<td>( X_3 = 1.0000 )</td>
<td>13.209 Net income year 2</td>
</tr>
<tr>
<td>( X_4 = 1.0000 )</td>
<td>13.209 Net income year 2</td>
</tr>
<tr>
<td>( X_5 = 0 )</td>
<td>0.0699 Sales growth year 1</td>
</tr>
<tr>
<td>( X_6 = 0.9504 )</td>
<td>0.0988 Sales growth year 2</td>
</tr>
<tr>
<td>( X_7 = 0 )</td>
<td>0.0988 Sales growth year 2</td>
</tr>
<tr>
<td>( X_8 = 0 )</td>
<td>70.129 NPV</td>
</tr>
<tr>
<td>( X_9 = 1.0000 )</td>
<td>70.129 NPV</td>
</tr>
</tbody>
</table>

Objective function 2:
The optimal LP solution showed 100% acceptance of projects 1, 3, 4, and 9, as well as 97% acceptance of project 6 and 4.5% acceptance of project 7, which generated an NPV level of 70.273. All the GP solution above also accept 100% of projects 2, 3, 4, and 9. The only difference among the solutions is in the area of the partially accepted projects.

In the first GP solution (objective function 1) project 2 enters into the solution because of its contribution to the achievement of the new goals in the GP formulation.

The other GP solution is virtually the same as the LP solution. Greater variation in the optimal solution would probably have been found if more projects were under evaluation and/or greater diversity of goals were included in the formulation.

<table>
<thead>
<tr>
<th>Project acceptance</th>
<th>Goal levels achieved</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = 1.0000$</td>
<td>7.235</td>
</tr>
<tr>
<td>$X_2 = 0$</td>
<td></td>
</tr>
<tr>
<td>$X_3 = 1.0000$</td>
<td>13.194</td>
</tr>
<tr>
<td>$X_4 = 1.0000$</td>
<td></td>
</tr>
<tr>
<td>$X_5 = 0$</td>
<td>0.0702</td>
</tr>
<tr>
<td>$X_6 = 0.9697$</td>
<td></td>
</tr>
<tr>
<td>$X_7 = 0.0455$</td>
<td>0.0986</td>
</tr>
<tr>
<td>$X_8 = 0$</td>
<td></td>
</tr>
<tr>
<td>$X_9 = 1.0000$</td>
<td>70.273</td>
</tr>
</tbody>
</table>

Net income year 1
Net income year 2
Sales growth year 1
Sales growth year 2
NPV
Chapter 4

CAPITAL BUDGETING UNDER UNCERTAINTY.

4.1. Sensitivity analysis/break even analysis.

Uncertainty means that more things can happen than will happen. Therefore, whenever we are confronted with a cash-flow forecast, we should try to discover what else can happen. If we can identify the major uncertainties, we may find that it is worth undertaking some additional preliminary research that will confirm whether the project is worthwhile. And even if we decided that we have done all we can to resolve the uncertainties, we still want to be aware of the potential problems. We do not want to be caught by surprise if things go wrong; we want to be ready to take corrective action.

Companies try to identify the principal threats to a project's success. The simplest way to do it is to undertake a sensitivity analysis. In this case the manager considers in turn each of the determinants of the project's success and estimates how far the present value of the project would be altered by taking a very optimistic view of that variable.

Sensitivity analysis of this kind is easy, but it is not always helpful. Variables do not usually change one at a time. If costs are higher than we expect, it is a good bet that sales volumes will be lower.

Many companies try to cope with this problem by examining the effect on the project of alternative plausible combinations of variables. In other words, they will estimate the net present value of the project under different scenarios and compare this estimate with the base case.

In a sensitivity analysis we change variables one at a time: when we analyze scenarios, we look at a limited
number of alternative combinations of variables.

Sensitivity analysis boils down to expressing cash flows in terms of unknown variables and then calculating the consequences of misestimating the variables. It forces the manager to identify the underlying variables, indicates where additional information would be most useful, and helps to expose confused or inappropriate forecasts.

One drawback to sensitivity analysis is that it always gives somewhat ambiguous results. For example, what exactly does optimistic or pessimistic mean? The marketing department may be interpreting the term in a different way from the production department.

When we undertake a sensitivity analysis of a project we are asking how serious it would be if sales on costs turn out to be worse than we forecasted. Managers sometimes prefer to rephrase this question and ask how bad sales can get before the project begins to lose money. This is known as break even analysis.

The break-even point is located where the total cost curve intersects the total revenue curve, or equivalently where total revenue equals total cost. Conventional break-even analysis represents total revenue and total cost with straight lines. This assumes that output and sales can be increased without changing price (at least, the effect of price changes are not shown) and that the firm operates at the same efficiency at all levels. Thus, to increase profit it is necessary merely to increase the number of units sold.

Example:
Suppose a firm is planning to buy a numerical control lathe for 50,000. It is expected to be used for 5 years and have receipts of 15,000; 15,000; 15,000; 14,300 and 14,000 at the end of each year.

It is the company's policy to use a 12% minimum attractive rate of return and a straight line depreciation method.
Management wants to conduct a sensitivity analysis with
respect to error in the forecast of initial investment, lifetime, end of year receipt and minimum rate of return.

Solution:
The solution to this problem was solved by a program developed for this purpose. The discussion of this program and the solution to this problem can be seen in chapter 6.

4.2. Decision trees.

A technique that has been recommended to handle complex, sequential decisions over time is the use of decision trees.

A decision tree is a formal representation of available decision alternatives at various points through time which are followed by chance events that may occur with some probability. A ranking of the available decision alternatives is usually achieved by finding the expected returns of the alternatives, which merely requires multiplying the returns earned by each alternative for various chance events by the probability that the event will occur and summing over all possible events.

Managers may be tempted to think only about the first accept-reject decisions and to ignore the subsequent investment decisions that may be tied to it. But if subsequent investment decisions depend on those made today, then today's decision may depend on what we plan to do tomorrow.

Of course, it would be imprudent for the firm to immediately implement the project with the highest expected return. This optimal action with the unidimensional criterion cries out for further analysis. Namely we should investigate the following:

1. The degree to which the estimated probabilities of the various states of the economy would have to change for
the optimal solution to no longer be optimal.

2. The extent to which the estimated returns associated with the alternatives and states of the economy would have to change in order to shift the optimal decision.

3. The degree of risk associated with lack of alternatives.

4. The utility that the firm attaches to each of the returns for each of the states of the economy based on the firm's goals, risk, posture, risk-return preferences and so on.

Thus decision tree analysis is advocated as an initial step which requires further analysis rather than the final word in the firm's effort to maximize shareholder's expected utility.

Example:

A firm is considering three alternative single-period investments, A, B, and C, whose returns are dependent upon the state of the economy in the coming period. The state of the economy is known only by a probability distribution:

<table>
<thead>
<tr>
<th>State of the economy</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fair</td>
<td>0.25</td>
</tr>
<tr>
<td>Good</td>
<td>0.40</td>
</tr>
<tr>
<td>Very good</td>
<td>0.30</td>
</tr>
<tr>
<td>Super</td>
<td>0.05</td>
</tr>
</tbody>
</table>

1.00

The returns for each alternative under each possible state of the economy are as follows:
The decision tree for this problem is shown below. Square nodes represent decision alternatives and round nodes show chance events.

Decision alternative the economy probability of state of the earned return weighted
State of the economy economy state of the economy

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Fair</th>
<th>Good</th>
<th>very good</th>
<th>super</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10</td>
<td>40</td>
<td>70</td>
<td>90</td>
</tr>
<tr>
<td>B</td>
<td>-20</td>
<td>50</td>
<td>100</td>
<td>140</td>
</tr>
<tr>
<td>C</td>
<td>-75</td>
<td>60</td>
<td>120</td>
<td>200</td>
</tr>
</tbody>
</table>

\[ E(R) = 44.00 \]

\[ E(R) = 52.00 \]

\[ E(R) = 51.25 \]
<table>
<thead>
<tr>
<th>Decision alternative</th>
<th>Expected return</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>44.00</td>
</tr>
<tr>
<td>B</td>
<td>52.00</td>
</tr>
<tr>
<td>C</td>
<td>51.25</td>
</tr>
</tbody>
</table>

Thus, the best alternative is B.

4.3. Simulation.

'Simulation is the imitation of a real-world system by using a mathematical model which captures the critical operating characteristics of the system as it moves through time encountering random events.

Major uses of simulation models:
1. To determine improved operating conditions (i.e., system design).
2. To demonstrate how a proposed change in policy will work and or how the new policy compares with the existing policies (i.e., system analysis or sensitivity analysis).
3. To train operating personnel to make better decisions, to react to emergencies in a more efficient and effective manner, and to utilize different kinds of information (i.e., simulation games and heuristic programming).

Simulation in capital budgeting can be done by the following steps:
step 1: Modeling the project.
   The first step in any simulation is to give the computer a precise model of the project.
step 2: Specifying probabilities.
step 3: Simulate the cash flows.

A simulation model is composed of the following four
major elements:
1. Parameters, which are input variables specified by the decision maker which will be held constant over all simulation runs.
2. Exogenous variables, which are input variables outside the control of the decision maker and are subject to random variation – hence, the decision maker must specify a probability distribution which describes possible events that may occur and their associated likelihood of occurrence.
3. Endogenous variables, which are output or performance variables describing the operations of the system and how effectively the system achieved various goals as it encountered the random events mentioned above.
4. Identities and operating equations which are mathematical expressions making up the heart of the simulation model by showing how the endogenous variables are functionally related to the parameters and exogenous variables.

General flowchart of a simulation:

```
start

read in:
1. parameters
2. probability distributions of exogenous variables

DO
I=1 to max

generate values for all endogenous variables by using the identities and operating equations
```

A

B
compute values for all endogenous variables by using the identities and operating equations

gather statistics

perform statistical analysis
print output, and plot empirical distributions

stop

The focus of the simulation is to develop empirical distributions for each endogenous variable in order to describe how efficiently and effectively the system operates.

Simulation can be a very useful tool. The discipline of building a model of the project can in itself lead us to a deeper understanding of the project. And once we constructed our model, it is a simple matter to see how the outcomes would be affected by altering the scope of the project or the distribution of any of the variables. A marine engineer uses a tank to simulate the performance of alternative hull designs but knows that it is impossible to fully replicate the conditions that the ship will encounter. In the same way, the financial manager can learn a lot from laboratory tests but cannot hope to build a model that accurately captures all the uncertainties and interdependencies that really surround a project.
Example: Simulation for investment planning.

A firm is considering a 10 million extension to its processing plant. The estimated service life of the facility is 10 years. The management expects to be able to utilize 250,000 tons of processed material worth 510 per ton at an average cost of 435 per ton.
What is the return that the company can expect? What is the risk?

The key input factor management has decided to use are:
1. Market size.
2. Selling price.
4. Share of market.
5. Investment required.
7. Operating costs.
8. Fixed costs.

The nine input factors fall into three categories:
1. Market analyses:
   Included are market size, market growth rate, the firm's share of the market, and selling price. For a given combination of these factors sales revenue may be determined.

2. Investment cost analyses.
   Being tied to the limits of service-life and operating cost characteristics expected, these are subject to various kinds of error and uncertainty.

3. Operating and fixed costs.
   These also are subject to uncertainty, but perhaps the easiest to estimate.
Result of the market research:

<table>
<thead>
<tr>
<th>Key factor</th>
<th>Expected value</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. Market analyses:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Market size (in tons)</td>
<td>250,000</td>
<td>100,000 - 340,000</td>
</tr>
<tr>
<td>2. Selling price (in $/ton)</td>
<td>510</td>
<td>385 - 575</td>
</tr>
<tr>
<td>3. Market growth rate</td>
<td>3%</td>
<td>0 - 6%</td>
</tr>
<tr>
<td>4. Eventual share of market</td>
<td>12%</td>
<td>3% - 7%</td>
</tr>
<tr>
<td><strong>B. Investment cost analyses:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Total investment required (in millions)</td>
<td>$9.5</td>
<td>$7.0 - $10.5</td>
</tr>
<tr>
<td>2. Useful life of facilities (in years)</td>
<td>10</td>
<td>5 - 15</td>
</tr>
<tr>
<td>3. Residual value at 10 years (in millions)</td>
<td>4.5</td>
<td>3.5 - 5.0</td>
</tr>
<tr>
<td><strong>C. Other costs:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Operating cost (in $/ton)</td>
<td>435</td>
<td>370 - 545</td>
</tr>
<tr>
<td>2. Fixed costs (in thousands)</td>
<td>300</td>
<td>250 - 375</td>
</tr>
</tbody>
</table>

Range figures represent approximately 1% to 99% probabilities. That is, there is only a 1 in a 100 chance that the value actually achieved will be respectively greater or less than the range.
The next step is to determine the returns that will result from random combinations of the factors involved. This requires realistic restrictions, such as not allowing the total market to vary more than some reasonable amount from year to year. Of course, any method of rating the return which is suitable to the company may be used at this point, in the actual case management preferred discounted cash flow.

For one trial actually made in this case, 3,600 discounted cash flow calculations, each based on a selection of the nine input factors, were run in two minutes. The resulting rate-of-return probabilities were read out immediately and graphed in the following table:

**Uncertainty portrayal:**

<table>
<thead>
<tr>
<th>Percent returns</th>
<th>Probability of achieving at least the return shown</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>96.5%</td>
</tr>
<tr>
<td>5%</td>
<td>80.6%</td>
</tr>
<tr>
<td>10%</td>
<td>75.2%</td>
</tr>
<tr>
<td>15%</td>
<td>53.8%</td>
</tr>
<tr>
<td>20%</td>
<td>43.0%</td>
</tr>
<tr>
<td>25%</td>
<td>12.6%</td>
</tr>
<tr>
<td>30%</td>
<td>0%</td>
</tr>
</tbody>
</table>

**Risk profile:**

Chances that ROR will be achieved
Under the method using single expected values, management arrives only at hoped for expectation of 25.2% after taxes (which as we have seen, is wrong unless there is no variability in the various input factors - a highly unlikely event).
Chapter 5.

CAPITAL RATIONING UNDER UNCERTAINTY.

In this case mainly mathematical programming and simulation are used.

The major applications of mathematical programming models under condition of risk in the capital budgeting area have been in stochastic LP, chance constrained programming and quadratic programming under uncertainty.

5.1. Stochastic Linear Programming.

In stochastic linear programming (SLP) a LP model replaces the identities and operating equations of the simulation model and the two-stage process proceeds as follows. In stage 1, we first set a number of decision variables and consider that they will be fixed (just like "parameters" in a simulation model) for all subsequent observations of random events. In stage 2, random events are generated and these values plus the parameters from stage 1 are substituted into the LP model. The LP is solved which provides one empirical observation of the optimal value of the LP objective function and the optimal values of the decision variables. Next, we go back and repeat the process of generating random events and solving LP problems some desired numbers of times, thereby deriving a complete empirical distribution for the LP objective function. Finally, we compare this empirical distribution with other empirical distribution arrived at using different stage 1 decisions in order to ascertain that set of stage 1 decisions which optimizes the decision maker's utility function.
5.2. Chance-constrained programming.

The next major category which has seen capital budgeting application is that of chance-constrained programming (CCP). The approach of CCP is to maximize the expected value of the objective function subject to constraints that are allowed to be violated some given percentage of the time due to random variation in the system. Chance constraints are arrived at as follows. Consider the usual constraints of LP:

\[ \sum_{j=1}^{N} a_{ij} x_j \leq b_i \]

Owing to randomness in either the \( a \) coefficients or the \( b \) right hand side values, we show that the constraints do not have to be satisfied all the time by associating a probability statement with the following constraint:

\[ p \left\{ \sum_{j=1}^{N} a_{ij} x_j \leq b_i \right\} \geq \alpha_i \]

where:

- \( p \) = probability
- \( \alpha_i \) = minimum probability that the decision maker is willing to accept that a given constraint is satisfied.

In such constraints, if \( \alpha_i = 0.90 \), for example, this would mean that the decision maker requires that the constraint be satisfied at least 90% of the time and that he is willing to allow \( \sum a_{ij} x_j \) to exceed \( b_i \) up to 10% of the time.

5.3. Quadratic programming.

Quadratic programming (QP) is the mathematical programming model wherein a non-linear objective function is optimized subject to linear constraints. This model is
far easier to solve than the non-linear programming model, because the feasible region is convex. The convexity assures that a local optimal solution is also the global optimal solution. This greatly facilitates the optimization process since the feasible region for a non-linear model is not necessary convex.

5.4. An example by Salazar and Sen.

An interesting example that shows how capital rationing under uncertainty can be solved by a combination of mathematical programming techniques and simulation is the article written by Salazar and Sen.

Techniques of simulation and stochastic LP (using Weingartner's Basic Horizon model of capital budgeting) are employed to compute the expected return (with an associated measure of the risk involved) of different portfolios of projects. The manager can then select the portfolio which is closest to his personal risk return preference. The model provides a practical guide to management in the entire process of project search, portfolio generation, and portfolio evaluation which characterizes the capital budgeting decision.

Salazar and Sen incorporate two kinds of uncertainty into their model: uncertainty related to significant economic and competitive variables which are likely to affect project cash flows and uncertainty related to the cash flows of the projects under consideration based on these variables.

Salazar and Sen handle the first type of uncertainty by a tree diagram similar to those introduced in the preceding chapter, shown in figure 5.4. There are 12 branches in the tree diagram with their respective joint probabilities of occurrence shown on the far right of the tree, the derivation of those probabilities is based on the
table below the tree. In the stochastic LP framework, the branches in this tree diagram will be considered as stage decisions that will be fixed; for each branch in the tree cash flows for each project under consideration will be randomly generated and then plugged into the LP model.

To elaborate, consider the flowchart used by Salazar and Sen which is shown in figure 5.5. The first processor box in the flowchart randomly selects a branch from the tree shown in figure 5.4. Next, the time counter, t, set to 1 and then the project counter, j, is also set to 1. The two Do loops randomly generated the cash flows for all project (u to j=J) over all time periods (up to t=T - the planning horizon of the model). These random cash flows are plugged into the model's LP algorithm and the optimal set of projects, and the optimal objective function value is obtained.

We then check to see if we have performed the desired number of simulations (S^*). If not, we get back and randomly select another branch from the tree diagram in figure 5.4 and repeat the simulation and LP solution again. When all simulation runs have been performed, the empirical LP objective function are plotted on the risk return axes (see figure 5.6).

Given this summary of the results, management has to decide which portfolio of assets optimized its utility function.
FIGURE 5.4. Tree Diagram Used by Salazar and Sen

Structure of the Model

<table>
<thead>
<tr>
<th>Time Period</th>
<th>Branch</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>0.072</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>0.168</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>0.216</td>
</tr>
<tr>
<td>4</td>
<td>D</td>
<td>0.144</td>
</tr>
<tr>
<td>5</td>
<td>E</td>
<td>0.020</td>
</tr>
<tr>
<td>6</td>
<td>F</td>
<td>0.080</td>
</tr>
<tr>
<td>7</td>
<td>G</td>
<td>0.050</td>
</tr>
<tr>
<td>8</td>
<td>H</td>
<td>0.016</td>
</tr>
<tr>
<td>9</td>
<td>I</td>
<td>0.144</td>
</tr>
<tr>
<td>10</td>
<td>J</td>
<td>0.012</td>
</tr>
<tr>
<td>11</td>
<td>K</td>
<td>0.028</td>
</tr>
<tr>
<td>12</td>
<td>L</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Interpretation of Chance Nodes

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Time Period</th>
<th>Environmental State</th>
<th>Name</th>
<th>Symbol</th>
<th>Probability of Occurrence</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNP</td>
<td>A</td>
<td>3</td>
<td>Increases</td>
<td>0</td>
<td>0</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>No change</td>
<td>1</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Fails</td>
<td>-1</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>(Competitor's price — our price)</td>
<td>B</td>
<td>5</td>
<td>(0 or -)</td>
<td>1</td>
<td>0</td>
<td>0.4</td>
</tr>
<tr>
<td>Introduction of new product by competitor</td>
<td>C</td>
<td>7</td>
<td>Yes</td>
<td>1</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>No</td>
<td>0</td>
<td>0</td>
<td>0.7</td>
</tr>
</tbody>
</table>

FIGURE 5.5 Macro Flowchart of System Program

Start

\[ S = 1 \]

Select Branch i (SIMBRANCH)

\[ i = 1 \]

Put Cash Flows in LP Tableau

Solve LP Problem (MINIT)

\[ S = S + 1 \]

Yes

Is \[ S > S^* \]?

No

\[ i = i + 1 \]

Select Cash Flow (NORMRAN)

\[ i = i + 1 \]

Is \[ i > J \]?

No

Yes

\[ r = r + 1 \]

Is \[ r > T \]?

End

The main advantages of Salazar and Sen model are:

1. It is conceptually simpler for the operating manager.
2. It generates a wider range of information in a variety of formats for management use.

It provides a practical guide to management in the entire process of project search, portfolio generation and the analysis and evaluation of alternative portfolios.
Chapter 6.

DESCRIPTION OF THE SENSITIVITY ANALYSIS PROGRAM.

In the capital budgeting problem, we usually make predictions about the profitability of an investment. We need to know how the profitability would be affected if an error should occur in the forecast.

This program calculates the annual worth (= net present value multiply by capital recovery factor) of an investment proposal and examines the impact of several forecast errors on the profitability.

Four types of errors are considered: error in the lifetime, the initial investment, the discount rate (MARR) and the end of year receipts. As such, the program provides information to make a more conscious decisions on the profitability of a project.

6.1. Assumptions.
- Depreciation method is linear.
- The effect of inflation is not considered.
- No expense.
- No taxes considered.
- Not assume a zero salvage value for any year after the year of the initial investment.

6.2. Input data.
1. Initial investment.
2. Life time (years)
3. Minimum attractive rate of return (percent)
4. Salvage value.
5. End of year receipts (each year during the life time)
6.3. Flowchart of the program.

- **Start**
- Input: Initial investment, Life time, MARR, Salvage value
- Input: End of year receipts
- Computation of annual worth with error in MARR for each point within the range
- Range = -99 to 99
- **Range = 99?**
  - **Y**
    - **NN = 1 to 2**
    - **NN = 1** = minimum value
    - **NN = 2** = maximum value
    - Computing minimum and maximum value of annual worth with error in annual receipts
    - Computing minimum and maximum value of annual worth with error in initial investment
    - **NN = 2?**
      - **Y**
      - A
Computing annual worth error in life time

Make a scaling for each type of annual worth error from its minimum and maximum values

Chained to a program named: Grafica

Drawing X axis and Y axis

Drawing dots for scaling

Drawing the line of annual worth with error in MARR

Drawing the line of annual worth with error in initial investment

Drawing the line of annual worth with error in annual receipts
B

Drawing the line of annual worth with error in lifetime

Drawing -50% and 50% signs at the right hand side and left hand side of the X axis

Drawing four descriptive lines (I, II, III, and III)

Printing text on screen

Chained to a program named Graph

Printing text on printer

Chained to a program named Printing

Printing curve on printer

Printing the result of the computation

END
6.4. Output data.

- Annual worth with 0% error.
- Annual worth scale.
- Graphic of annual worth (X axis) vs % error (Y axis):
  I = years curve
  II = initial investment curve
  III = MARR curve
  IIII = End of year receipt curve

- Values of annual worth for an error in:
  - MARR
  - Initial investment
  - EOY receipts
- Values of annual worth for an error in life time.

6.5. Example:
The example that will be used here is taken from chapter 4. (text about sensitivity analysis).
The solution with the program is shown below

Input:
1. Initial investment = 50,000
2. Life time = 5 years
3. MARR = 12%
4. Salvage value = 0
5. End of year receipts: year 1 = 15,000
   2 = 15,000
   3 = 15,000
   4 = 14,500
   5 = 14,000
Result:

DETERMINISTIC SENSITIVITY ANALYSIS

NET ANNUAL WORTH ($) VS. %-ERROR
(X-AXIS) (Y-AXIS)

I = YEARS CURVE
II = INITIAL INVESTMENT CURVE
III = MARR CURVE
III = EDY RECEIPT CURVE

ANNUAL WORTH (% ERROR): $863.95
ANNUAL WORTH SCALE: $4000
DETERMINISTIC SENSITIVITY ANALYSIS

<table>
<thead>
<tr>
<th>Initial</th>
<th>EDY</th>
<th>MARR</th>
<th>Investment</th>
<th>Receipts</th>
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<td>-150</td>
<td>4522.27</td>
<td>14060.92</td>
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**Range:** 7711.2 * 26553.93 * 26553.44
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In this study we have examined the techniques used to evaluate capital investments.

For capital budgeting problems under certainty we explore five alternatives, namely net present value, payback, average return on book value, internal rate of return and profitability index.

We also examined the use of mathematical programming (linear programming, integer programming and goal programming) for the capital rationing problems under certainty. The aim of these techniques is to find the combination of projects that will maximize shareholder's wealth while not violating any budget constraint. We have also examined Weingartner's basic horizon model for this problem.

For the capital budgeting under uncertainty we explored the use of sensitivity analysis/break even analysis, decision trees and simulation.

For the capital rationing under uncertainty our survey included stochastic linear programming, chance-constrained programming and quadratic programming. These techniques will determine the set of projects that will maximize expected shareholder's utility under the complex conditions of multi-period uncertainty.

We tried to use and to develop computer programs that can be applied for capital budgeting/rationing problem.
REFERENCES.


APPENDIX

1 REM -----------------------------
2 REM DETERMINISTIC SENSITIVITY
3 REM -----------------------------
4 HOME
5 TI$ = "SENSITIVITY ANALYSIS"
6 T2$ = "FOR"
7 T3$ = "AN ECONOMIC EVALUATION"
8 T4$ = "OF"
9 T5$ = "CAPITAL PROJECTS"
10 VTAB 7: HTAB 10
11 FOR N = 1 TO LEN (TI$): PRINT MID$ (TI$,N,1));
12 FOR I = 1 TO 50: NEXT I: NEXT N: PRINT : PRINT : HTAB 18
13 FOR N = 1 TO LEN (T2$): PRINT MID$ (T2$,N,1));
14 FOR I = 1 TO 50: NEXT I: NEXT N: PRINT : PRINT : HTAB 9
15 FOR N = 1 TO LEN (T3$): PRINT MID$ (T3$,N,1));
16 FOR I = 1 TO 50: NEXT I: NEXT N: PRINT : PRINT : HTAB 19
17 FOR N = 1 TO LEN (T4$): PRINT MID$ (T4$,N,1));
18 FOR I = 1 TO 50: NEXT I: NEXT N: PRINT : PRINT : HTAB 12
19 FOR N = 1 TO LEN (T5$): PRINT MID$ (T5$,N,1));
20 FOR I = 1 TO 50: NEXT I
21 HOME: VTAB 2: HTAB 9: PRINT "ENTER THE FOLLOWING VALUES"
22 POKE 34,3
23 VTAB 10: INPUT "INITIAL INVESTMENT : $";IN
25 VTAB 13: INPUT "LIFE TIME (YEARS) : ";YR
26 VTAB 16: INPUT "MARR (PERCENT) : ";TE
27 VTAB 19: INPUT "SALVAGE VALUE : ";SA
28 HOME: HTAB 9: PRINT "EDY"; TAB (28); "RECEIPT"
29 HTAB 9: PRINT "---"; TAB (28); "----"
30 DIM AW (200) , B (200) , MA (200) , A (200)
31 FOR G = 1 TO YR: HTAB 10
32 PRINT G: HTAB 27: INPUT "$",A(G): NEXT G
33 POKE 34,0
34 HOME: VTAB 12: HTAB 16: FLASH: PRINT "COMPUTING"; NORMAL
35 I = TE / 100
36 FOR R = - 99 TO 99 : P = 1 + R / 100: Z = 0
37 FOR N = 1 TO YR: Z = Z + A(N) * (1 + I * P) ^ (- N): NEXT N
38 QD = 1 + I * P: PW = - N + Z + SAL * QQ ^ (- YR)
39 AW (R + 100) = PW * ((I * P * QQ ^ YR) / (QQ ^ YR - 1))
40 NEXT R
41 FOR NN = 1 TO 2: IF NN = 1 THEN P = 1 - 99 / 100
42 IF NN = 2 THEN P = 1 + 99 / 100
43 Z = 0
44 FOR N = 1 TO YR: Z = Z + A(N) * P * (1 + I) ^ (- N): NEXT N
45 PW = - IN + Z + SAL * (1 + I) ^ (- YR)
46 AA (NN) = PW * ((I * (1 - I) ^ YR) / ((1 + I) ^ YR - 1)): Z = 0
47 FOR N = 1 TO YR: Z = Z + A(N) * (1 + I) ^ (- N): NEXT N
48 PW = - IN * P + SAL * (1 + I) ^ (- YR) + Z
49 WW (NN) = PW * ((I * (1 - I) ^ YR) / ((1 + I) ^ YR - 1))
50 NEXT NN
52 Z = 0:L = 2: YR - 1: C = YR + 1: XY = 0: X = 0: Y = 0: XSD = 0
53 FOR S = 1 TO YR
54 XY = XY + S: A(S):X = X + S: Y = Y + A(S): XSQ = XSQ + S * 2
55 NEXT S
56 B = (YR * XY - X * Y) / (YR * XSQ - X * 2)
57 A = Y / YR - B * X
58 DEF FN F(X) = A + B * X
59 DEF FN SAL(X) = (SAL - IN) / YR * X + IN
60 FOR N = C TO L: A(N) = FN F(N): NEXT N
61 FOR N = 1 TO L
62 Z = Z + A(N) * (1 + I) ^ (- N)
63 IF FN SAL(N) < 0 THEN GOSUB 88: GOTO 67
64 PW = - IN + Z + FN SAL(N) * (1 + I) ^ (- N)
65 MA(N) = PW * ((1 + (1 + I) ^ N) / ((1 + I) ^ N - 1))
66 B(N) = (100 * N) / YR - 100
67 NEXT N
68 HOME: VTab 12: HTab 17: FLASH: PRINT "SCALING": NORMAL
69 K(1) = ABS (AW(1)): K(2) = ABS (AW(199)): K(3) = ABS (TT)
70 K(4) = ABS (TU): K(5) = ABS (UT): K(6) = ABS (ST)
71 K(7) = ABS (MA(L - 1)): MAX = K(1)
72 FOR M = 2 TO 8
73 IF K(M) > MAX THEN MAX = K(M)
74 NEXT M: SC = INT (MAX / 5)
75 FOR U = 0 TO 900 STEP 100
76 IF SC > U AND SC < = U + 100 THEN SC = U + 100: GOTO 36
77 NEXT U
78 L(1) = 1000: L(2) = 1E4: L(3) = 1E5: L(4) = 1E6: L(5) = 1E7
79 U(1) = 9000: U(2) = 9E4: U(3) = 9E5: U(4) = 9E6: U(5) = 9E7
80 FOR S = 1 TO 5
81 FOR U = L(S) TO U(S) STEP L(S)
82 IF SC > U AND SC < = U + L(S) THEN SC = U + L(S): GOTO 35
83 NEXT U
84 NEXT S
85 END
86 PRINT CHR$(4); "SLOAD CHAIN, A520"
87 CALL 520 "GRAPICA"
88 PW = - IN + Z: B(N) = (100 * N) / YR - 100
89 MA(N) = PW * ((1 + (1 + I) ^ N) / ((1 + I) ^ N - 1)): RETURN
90 REM -----------------------
91 REM GRAFICA
92 REM -----------------------
93 DEF FN S(X) = INT (((9 + 14 * (5 + SC - X) / SC) + .5)
94 DEF FN G(X) = INT ((15.5 + 252 / 190 * (95 + X))
95 HGR ; HCOLOR= 6
96 HPLT 140,0 TO 140,159: HPLT 0,79 TO 279,79
97 FOR N = 14 TO 266 STEP 14: HPLT N,79 TO N,77: NEXT N
98 FOR N = 9 TO 149 STEP 14: HPLT 139,N TO 142,N: NEXT N
99 FOR N = 14 TO 210 STEP 14
100 FOR V = 2 TO 159 STEP 7
101 HPLT N,V
102 NEXT V
103 NEXT N
104 FOR N = 224 TO 266 STEP 14
105 FOR V = 2 TO 114 STEP 7
106 HPLT N,V
107 NEXT V
108 NEXT N
109 HCOLOR= 2
110 FOR N = -99 TO 98: DE = AW(N + 100): RA = AW(N + 101)
111 HPLT FN G(N), FN S(DE) TO FN G(N + 1), FN S(RA)
112 NEXT N
113 HCOLOR= 5: HPLT FN G( -99), FN S(TU) TO FN G(99), FN S(ST)
114 HCOLOR= 1: HPLT FN G( -99), FN S(SU) TO FN G(99), FN S(TT)
115 HCOLOR= 7: R = 2 * YR - 2
116 FOR N = 1 TO R: SS = B(N + 1): SR = MA(N + 1)
117 HPLT FN G(S(N)), FN S(MA(N)) TO FN G(SS), FN S(SR)
118 NEXT N
119 HCOLOR= 6: HPLT 59,54 TO 61,84
120 FOR N = 82 TO 94 STEP 2: HPLT 63,N TO 66,N
121 HPLT 204,N TO 206,N: NEXT N
122 HPLT 63,87 TO 66,87: HPLT 204,87 TO 206,87: HPLT 64,83
123 HPLT 204,83: HPLT 66,85: HPLT 206,85: HPLT 66,86
124 HPLT 206,86: HPLT 70,82 TO 70,87: HPLT 210,82 TO 210,87
125 HPLT 74,82 TO 74,87: HPLT 214,82 TO 214,87
126 HPLT 72,82: HPLT 212,82: HPLT 72,87: HPLT 212,87
127 HPLT 66,96 TO 72,90: HPLT 206,96 TO 212,90
128 FOR N = 90 TO 91: HPLT 66,N: HPLT 206,N
129 HPLT 64,N: HPLT 204,N: NEXT N
130 FOR N = 95 TO 96: HPLT 74,N: HPLT 214,N
131 HPLT 72,N: HPLT 212,N: NEXT N
132 HCOLOR= 7: HPLT 224,128 TO 252,128
133 HCOLOR= 6: HPLT 266,125 TO 266,129
134 HCOLOR= 5: HPLT 266,135 TO 252,135
135 HCOLOR= 6: HPLT 266,132 TO 264,136: HPLT 268,132 TO 268,136
136 HCOLOR= 2: HPLT 224,142 TO 222,142
137 HCOLOR= 6: HPLT 266,139 TO 266,143: HPLT 262,139 TO 262,143
138 HPLT 270,139 TO 270,143
139 HCOLOR= 1: HPLT 224,149 TO 252,149: HCOLOR= 6
140 FOR N = 260 TO 272 STEP 4: HPLT N,146 TO N,150: NEXT N
141 HOME : VTAB 21
142 PRINT TAB(2)"ANNUAL WORTH (Y) VS % EST.ERROR (X)"
143 PRINT TAB(2)"AW SCALE : $; SC:$TAB(33)"I=YEARS"
144 PRINT TAB(2)"II=INIT INVEST III=MARR IIII=RECEIPTS"
145 HTAB 8: PRINT "TYPE ANY KEY TO CONTINUE": GET K$" 
146 HOME : VTAB 13: HTAB 15: FLASH : PRINT "PLEASE WAIT": NORMAL
147 PRINT CHR$ (4); "LOAD CHAIN,AS20"
148 CALL 520"GRAPH"
149 REM __________________________
150 REM GRAPH
151 REM __________________________
152 HOME: VTab 22: HTab 17: FLASH: PRINT "PRINTING": NORMAL
153 PRINT CHR$ (4); "PR#1"
154 PRINT CHR$ (09) + "801"
155 PRINT : PRINT : PRINT : PRINT : PRINT
156 PRINT TAB( 24) "DETERMINISTIC SENSITIVITY ANALYSIS"
157 PRINT TAB( 24) "XXXXXXXXXXXXXXXXXXXXXXXXX"
158 FOR S = 1 TO 6: PRINT : NEXT S
159 PRINT TAB( 25) "NET ANNUAL WORTH($) VS. %-ERROR"
160 PRINT TAB( 31) "(X-AXIS)
161 PRINT : PRINT : PRINT TAB( 27) "I=YEARS CURVE"
162 PRINT : PRINT TAB( 27) "II=INITIAL INVESTMENT CURVE"
163 PRINT : PRINT TAB( 27) "III=MARR CURVE"
164 PRINT : PRINT TAB( 27) "IV=EDY RECEIPT CURVE"
165 FOR J = 1 TO 10: PRINT : NEXT J
166 CU = INT (AW(100) * 100 + .5) / 100
167 PRINT TAB( 27) "ANNUAL WORTH(%ERROR): $"; CU: PRINT
168 PRINT TAB( 27) "ANNUAL WORTH SCALE : $"; SC
169 PRINT CHR$ (09) + "9"
170 PRINT CHR$ (4) "PR#0": TEXT: HOME: VTab 12: HTab 14
171 FLASH: PRINT "PLEASE WAIT": NORMAL
172 PRINT CHR$ (4) "SLOAD CHAIN,A520"
173 CALL S20"PRINTING"
174 REM ----------------------------------
175 REM PRINTING
176 REM ----------------------------------

177 HOME : VTAB 12: HTAB 16: FLASH : PRINT "PRINTING": NORMAL
178 PRINT CHR$ (4); "PR#1"
179 PRINT CHR$ (09) + "80N"

180 FOR Q = 1 TO 5: PRINT : NEXT Q
181 PRINT TAB( 24)"DETERMINISTIC SENSITIVITY ANALYSIS"
182 PRINT TAB( 24)"********************************************************************" 
183 FOR N = 1 TO 4: PRINT : NEXT N
184 HTAB 20: PRINT "********************************************************************"
185 HTAB 20: PRINT "ANNUAL WORTH FOR AN ERROR IN:
186 HTAB 20: PRINT "**\% INITIAL **\% EDY **\%"
187 HTAB 20: PRINT "ERROR ** MARR **INVESTMENT ** RECEIPTS **"
188 HTAB 20: PRINT "********************************************************************"
189 HTAB 20: PRINT "********************************************************************"
190 FOR MM = 5 TO 195 STEP 5
191 AW(MM) = INT (AW(MM) * 100 + .5) / 100: NEXT MM
192 K = TT - TU: S = 198: C = 100: E = ST - UT: J = 1
193 DEF FN EDY(X) = INT ((K / S * (X + 99) + TU) * C + .5) / C
194 DEF FN IN(X) = INT ((E / S * (X + 99) + UT) * C + .5) / C
195 FOR N = -95 TO 95 STEP 5
196 HTAB 20: PRINT "**********
197 PRINT SPC( 9 - LEN (STR$(N))): $N; " **
198 PRINT SPC( 9 - LEN (STR$(AW(N + 100)))): AW(N + 100); " **
199 PRINT SPC( 9 - LEN (STR$(FN IN(N)))): FN IN(N); " **
200 PRINT SPC( 9 - LEN (STR$(FN EDY(N)))): FN EDY(N); " 
201 NEXT N
202 HTAB 20: PRINT ".
203 HTAB 20: PRINT "********************************************************************"
204 L = 2 + YR - 1: MA(L) = INT (MA(L) * 100 + .5) / 100
205 A = ABS( (AW(5) - AW(195) )
206 C = ABS( (FN IN(-95) - FN IN(95))
207 D = ABS( (FN EDY(95) - FN EDY(-95))
208 D = INT (D * 100 + .5) / 100
209 HTAB 20: PRINT "RANGE**
210 PRINT SPC( 9 - LEN (STR$(A))): A**
211 PRINT SPC( 9 - LEN (STR$(C))): C**
212 HTAB 20: PRINT "********************************************************************
213 FOR R = 1 TO 5: PRINT : NEXT R
214 HTAB 31: PRINT "ERROR IN LIFE TIME"
215 HTAB 31: PRINT "*----------------------*
216 HTAB 30: PRINT "EDY ANNUAL WORTH"
217 HTAB 30: PRINT "*----------------------*
218 FOR N = 1 TO L: MA(N) = INT (MA(N) * 100 + .5) / 100
219 HTAB 31: PRINT SPC( 16 - LEN (STR$(MA(N))): MA(N)
220 PRINT SPC( 16 - LEN (STR$(MA(N))): MA(N)
221 NEXT N
222 PRINT CHR$ (4); "PR#0": HOME
223 END